Economics 662

October 21, 2025 **R. Davidson**

Midterm Examination

Please send your completed exam to Miroslav Zhao <**miroslav.zhao@mail.mcgill.ca**>*, or upload it to myCourses, by 12.00 (noon) on Friday October 24. As for the assignments, please submit two files per student: one, which should be a PDF file, with your written answers, and another, which may or may not be a simple text file, with your computer code. These files must be all your own work. You may make use of whatever non-human resources you wish, provided that you acknowledge them with a suitable citation, but you must not ask for or receive any help from any other person. Note that the questions all have different weights in the marking scheme.*

All students in this course have the right to submit in English or in French any written work that is to be graded.

Tou(te)s les étudiant(e)s qui suivent ce cours on le droit de soumettre tout travail écrit en français ou en anglais.

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Answer all five questions in this exam.

Faites tous les cinq exercices de cet examen.

1. A square matrix $\boldsymbol{U}$ is said to be **orthogonal** if its transpose is equal to its inverse: $\boldsymbol{U}^\top\boldsymbol{U} = \boldsymbol{U}\boldsymbol{U}^\top = \mathbf{I}$. Show that the product of two orthogonal matrices of the same dimensions is also an orthogonal matrix.

Let $\boldsymbol{P_X}$ be the orthogonal projection matrix of which the image is the linear span of the k columns of the $n \times k$ matrix $\boldsymbol{X}$, where $n > k$. What are the defining properties of any orthogonal projection matrix? Give an explicit expression for the matrix $\boldsymbol{P_X}$ in terms of $\boldsymbol{X}$.

Now let $\boldsymbol{M_X} \equiv \mathbf{I} - \boldsymbol{P_X}$ be the complementary projection matrix. Show that the product $\boldsymbol{P_X}(\mathbf{I} - \boldsymbol{P_X})$ is a zero matrix *without* making use of the fact that $\boldsymbol{P_X}$ is an orthogonal projection matrix rather than any projection matrix. If $\boldsymbol{P_X}$ is in fact an orthogonal projection matrix, prove that $\boldsymbol{M_X}$ is so also, making use in your proof of the defining properties.

Show that the matrix $\boldsymbol{M_X} - \boldsymbol{P_X}$ is an orthogonal matrix. What is its inverse?

2. State the Frisch-Waugh-Lovell (FWL) Theorem. Explain how to use it to generate deseasonalised variables that can then be used in linear regression models.

In each of the following regressions, y_t is the dependent variable, x_t and z_t are exogenous explanatory variables, and α, β, and γ are unknown parameters.

a) $y_t = \alpha + \beta x_t + \gamma/x_t + u_t$
b) $y_t = \alpha + \beta x_t + x_t/\gamma + u_t$
c) $y_t = \alpha + \beta x_t + z_t/\gamma + u_t$
d) $y_t = \alpha + \beta x_t + z_t/\beta + u_t$
e) $y_t = \alpha + \beta x_t z_t + u_t$
f) $y_t = \alpha + \beta\gamma x_t z_t + \gamma z_t + u_t$
g) $y_t = \alpha + \beta\gamma x_t + \gamma z_t + u_t$
h) $y_t = \alpha + \beta x_t + \beta x_t^2 + u_t$
i) $y_t = \alpha + \beta x_t + \gamma x_t^2 + u_t$
j) $y_t = \alpha + \beta\gamma x_t^3 + u_t$
k) $y_t = \alpha + \beta x_t + (1 - \beta)z_t + u_t$
l) $y_t = \alpha + \beta x_t + (\gamma - \beta)z_t + u_t$

For each regression, say whether it is possible to obtain an estimator of the parameters by least squares. In other words, are the parameters identified? If not, explain why not. If so, can the estimator be obtained by *ordinary*, that is, linear, least squares? If so, give the explicit expression of the regressand and regressors to use in the OLS regression.

3. Show that the covariance of the random variable $\mathrm{E}(X_1 \mid X_2)$ and the random variable $X_1 - \mathrm{E}(X_1 \mid X_2)$ is zero. It is easiest to show this result by first showing that it is true when the covariance is computed conditional on X_2.

Show that the variance of the random variable $X_1 - \mathrm{E}(X_1 \mid X_2)$ cannot be greater than the variance of X_1, and that the two variances are equal if X_1 and X_2 are independent. This result shows how one random variable can be informative about another: Conditioning on it reduces variance unless the two variables are independent.

Let a random variable X_1 be distributed as $\mathrm{N}(0, 1)$. Now suppose that a second random variable, X_2, is constructed as the product of X_1 and an independent random variable Z, which equals 1 with probability $1/2$ and -1 with probability $1/2$.

What is the (marginal) distribution of X_2? What is the covariance between X_1 and X_2? What is the distribution of X_1 conditional on X_2?

4. The **Cauchy distribution** is defined as the distribution of the ratio of two independent standard normal variables, Z_1 and Z_2. Thus the random variable $W = Z_1/Z_2$ has the Cauchy distribution. Show that $W' = Z_1/|Z_2|$ also has the Cauchy distribution.

Show that the density of the Cauchy distribution is

$$f(w) = \frac{1}{\pi(w^2 + 1)}.$$

Carry out a simulation experiment in which you generate a large number N of IID realisations from the Cauchy distribution, W_i, $i = 1, \ldots, N$, and for $n = N/10, \ldots, N$, set $Y_n = n^{-1} \sum_{i=1}^{n} W_i$. Plot Y_n as a function of n. Since the Cauchy distribution has no moments, the law of large numbers should not apply to the sequence $\{Y_n\}$. Does your plot reflect this fact?

Repeat the above exercise with realisations from the standard normal distribution. Now, the law of large numbers should indeed apply. Does your plot reflect this fact this time?

On to the central-limit theorem. For a sample size n of realisations x_i, $i = 1, \ldots, n$, from $\mathrm{N}(0, 1)$, the theorem concerns the distribution of the weighted sums

$$z = n^{-1/2} \sum_{i=1}^{n} x_i.$$

Generate a large number N of weighted sums z_j, $j = 1, \ldots, N$, and plot the empirical distribution of the z_i along with the CDF Φ of the standard normal distribution. Does your simulation seem in accord with the central-limit theorem?

5. Give the formal definition of a positive definite matrix. How is this definition different from that of a positive semi-definite matrix? If $\boldsymbol{A}$ is positive definite, show that $\boldsymbol{A}^{-1}$ is so also. Why does the same result not hold in general for positive semi-definite matrices?

Suppose that we wish to carry out a two-tailed test using a test statistic τ which has a distribution that is not symmetric about the origin. Let the CDF of the statistic τ be denoted as F, where $F(-x) \neq 1 - F(x)$ for general x. Suppose that, for any significance level α, the critical values c_α^- and c_α^+ are defined by the equations

$$F(c_\alpha^-) = \alpha/2 \quad \text{and} \quad F(c_\alpha^+) = 1 - \alpha/2.$$

Show that the marginal significance level (P value) associated with a realized statistic $\hat{\tau}$ is $2\min\bigl(F(\hat{\tau}), 1 - F(\hat{\tau})\bigr)$.