A Construction of Brownian Motion as an Almost-Sure Limit

1. Introduction

If $\{z_t\}$ is a sequence of IID random variables with $\mathrm{E}(z_t) = 0$ and $\mathrm{E}(z_t^2) = 1$, then the central limit theorem tells us that the sequence with typical element

$$Z_n \equiv n^{-1/2} \sum_{t=1}^{n} z_t \tag{1}$$

is asymptotically normal. Specifically, the sequence $\{Z_n\}$ tends in distribution to the standard normal distribution:

$$Z_n \xrightarrow[D]{} \mathrm{N}(0,1).$$

We may define a stochastic process on the $[0,1]$ interval by means of the sequence $\{z_t\}$ as follows:

$$W^n(t) = n^{-1/2} \sum_{t=1}^{\lfloor nt \rfloor} z_t, \quad t \in [0,1]. \tag{2}$$

Then the functional central limit theorem tells us that, as $n \to \infty$,

$$W^n(t) \xrightarrow[D]{} W(t),$$

where $W(t)$ is a standard Wiener process, or Brownian motion, on $[0,1]$.

A difficulty with this construction is that the limit is *only* in distribution, and not in probability, still less almost sure. To see this, suppose that $Z_n \to Z$ in probability, where Z is some random variable. Then we show that Z and the summands z_t are independent. Note that, for the given t,

$$Z_{tn} \equiv n^{-1/2} \sum_{s=t+1}^{t+n} z_s$$

also tends to N(0,1) in distribution as $n \to \infty$. Under the assumption that $Z_n \to Z$ in probability, it follows also that $Z_{tn} \to Z$ in probability. Consider the joint characteristic function of z_t and Z. It is, for arbitrary real arguments s and r,

$$\begin{aligned}
\mathrm{E}\exp(isZ + irz_t) &= \lim_{n\to\infty} \mathrm{E}\exp(isZ_{tn} + irz_t) \\
&= \lim_{n\to\infty} \mathrm{E}\exp(isZ_{tn})\mathrm{E}\exp(irz_t) \qquad (Z_{tn} \text{ is independent of } z_t) \\
&= \mathrm{E}\exp(isZ)\mathrm{E}\exp(irz_t).
\end{aligned}$$

The factorisation of the joint characteristic function demonstrates the independence of Z and z_t, for any t. Intuitively, the weight of z_t in the partial sums Z_n gets smaller as $n \to \infty$, and in the limit we have independence. A straightforward extension of this proof shows that Z_n is independent of Z for any n.

Now $Z \sim \mathrm{N}(0,1)$, and so $-Z \sim \mathrm{N}(0,1)$ as well. By independence, therefore, the joint distribution of Z_n and Z is the same as that of Z_n and $-Z$. Consequently,

$$\Pr(|Z_n - Z| > \varepsilon) = \Pr(|Z_n + Z| > \varepsilon)$$

for all n and all $\varepsilon > 0$. But this means that $Z_n \to -Z$ in probability, which is incompatible with $Z_n \to Z$ unless $Z = 0$. But that too is contradicted by the fact that $Z \sim \mathrm{N}(0,1)$, and so we conclude that the probability limit Z cannot exist.

2. A Different Construction

Despite the above result, it is possible to construct a sequence $\{Z_n\}$ of variables, where each Z_n has the same distribution as the partial sum (1) with IID summands, and $Z_n \to Z$ almost surely, with $Z \sim \mathrm{N}(0,1)$. The key is to fill in the terms of the partial sum from the middle, rather than continually appending new, independent, terms at the end.

We begin with the simplest case, in which the summands z_t are NID(0,1) themselves. As we will see, the construction provides as a by-product a means for simulating a Wiener process in continuous time directly. The starting point is to generate the realisation of $W(1)$, of which the marginal distribution is just N(0,1). At step i of the construction, we have a sequence z_{ti}, $t = 0, 1, \ldots, 2^i$, of NID(0,1) variables that define the stochastic process $W^i(t)$ through the formula (2) with $n = 2^i$. The process $W^i(t)$ is such that, for all $j < i$,

$$W^j(2^{-j}k) = W^i(2^{-j}k) \text{ for } k = 0, 1, \ldots 2^j.$$

This means that the values of the processes W^i at the dyadic points $2^{-j}k$, $k = 0, 1, \ldots, 2^j$, are the same for all i with $i \geq j$.

We can simplify notation by omitting obvious powers of 2. Thus, instead of $W^i(2^{-i}k)$, we may write simply $W^i(k)$ for $k = 0, 1, \ldots, 2^i$. To go from step i to step $i+1$, we must establish the values of W^{i+1} at the points $2^{-(i+1)}(2k+1)$, $k = 0, 1, \ldots, 2^i - 1$. These are the points midway between the points at which values are permanently established at step i, and, with them, constitute the set of points at which values are permanently established at step $i+1$. For the construction to be correct, it must be the case that $W^i(k) \sim \mathrm{N}(0, 2^{-i}k)$. In addition, we require that the increments $W^i(k+1) - W^i(k) \sim \mathrm{N}(0, 2^{-i})$, and that they are independent across k. Suppose that we have achieved these requirements at step i; we show how to maintain them at step $i+1$. Note that these requirements are enough for our claim that the distribution of

$$Z_{2^i} \equiv 2^{-i/2} \sum_{k=0}^{2^i - 1} 2^{i/2}\big(W^i(k+1) - W^i(k)\big)$$

is that of a sum of 2^i NID(0,1) variables, divided by $2^{i/2}$.

What then is the distribution of $W^{i+1}(2k+1)$ conditional on the $W^i(k)$, $k = 0, 1, \ldots, 2^i$? Since $W^i(0) = 0$ by construction, the conditioning is equivalent to conditioning on the increments $W^i(k+1) - W^i(k)$, $k = 0, 1, \ldots, 2^i - 1$. The conditional distribution is established if we can find that of the increment $W^{i+1}(2k+1) - W^{i+1}(2k)$, which is just $W^{i+1}(2k+1) - W^i(k)$, since the value at $2^{-i}k$ is permanently established at step i. Since we want both this increment and the next one, namely $W^i(k+1) - W^{i+1}(2k+1)$, to be independent of all other summands at step $i+1$, we see that it is enough to condition on $W^i(k+1) - W^i(k)$, which is independent of all the other step i increments.

Let

$$X_1 = 2^{i/2}\big(W^{i+1}(2k+1) - W^i(k)\big) \text{ and}$$
$$X_2 = 2^{i/2}\big(W^i(k+1) - W^{i+1}(2k+1)\big).$$

Then we require that X_1 and X_2 should be independent, each with marginal distribution $\mathrm{N}(0, 1/2)$, and such that $X_1 + X_2 = 2^{i/2}\big(W^i(k+1) - W^i(k)\big)$, of which the marginal distribution is N(0,1). The joint distribution of X_1 and $X \equiv X_1 + X_2$ is normal, with covariance matrix

$$\begin{bmatrix} 2^{-1} & 2^{-1} \\ 2^{-1} & 1 \end{bmatrix}.$$

The correlation is therefore $1/\sqrt{2}$. It follows that the expectation of X_1 conditional on X is $X/2$, and the conditional variance is $1/4$. Thus we may set $X_1 = \frac{1}{2}(X + U)$, where $U \sim \mathrm{N}(0, 1)$, independent of any random number used up to step i. We can check that, with this definition, $\mathrm{Var}(X_1) = 1/2$, as required. In addition $X_2 = X - X_1 = (X - U)/2$ has a variance of $1/2$. Further, the covariance of X_1 and X_2 is

$$\mathrm{cov}(X_1, X_2) = \tfrac{1}{4}\mathrm{E}\big((X+U)(X-U)\big) = \tfrac{1}{4}(1-1) = 0,$$

so that X_1 and X_2 are independent. In terms of the W^i, we have, for $k = 0, 1, \ldots, 2^i - 1$,

$$W^{i+1}(2k+1) = \tfrac{1}{2}\big(W^i(k) + W^i(k+1)\big) + 2^{-(i+2)/2}U_{i+1,k},$$

where we have indexed the innovation U so as to make it clear when it is used in the whole procedure.

We may denote by $\mathcal{F}_i$ the sigma-algebra generated by all the innovations used up to and including step i. Then $\mathcal{F}_0$ is generated by $U_{0,0} \sim \mathrm{N}(0, 1)$, and the process W^i is $\mathcal{F}_i$–measurable.

For any number $t \in [0, 1]$ with a finite dyadic expansion

$$t = \sum_{j=1}^{n} b_j 2^{-j}, \quad b_j = 0 \text{ or } 1,$$

for some finite n, it is clear that the sequence $\{W^i(t)\}$ converges to $W^n(t)$ as $i \to \infty$, since $W^i(t) = W^n(t)$ for all $i \geq n$. It remains to find the best argument to show that $\{W^i(t)\}$ converges almost surely for all real $t \in [0, 1]$.