Economics 765

May 9, 2024 **R. Davidson**

Assignment 1

You are asked to do exercises 1.2, 1.3, 1.6, and 1.11 of Volume 2 of Shreve. The essence of these exercises is reproduced below for convenience.

1.2 The infinite coin-toss space $\Omega_\infty$is *uncountably* infinite. That means that the sequences that are the elements of Ω_∞ cannot be put into a one-to-one correspondence with the positive integers. Shreve provides the **diagonal argument** that proves this statement. Then Shreve asks:

Now consider the set A of coin tosses in which the outcome on each even-numbered toss matches the outcome of the toss preceding it, that is

$$A = \{\omega = \omega_1\omega_2\omega_3\omega_4\ldots : \omega_2 = \omega_1; \omega_4 = \omega_3, \ldots\}.$$

(i) Show that A is uncountably infinite;

(ii) Show that when p, the probability of heads on any toss, is in the interior of the interval $(0, 1)$, then $\mathbb{P}(A) = 0$.

We are then told that an uncountably infinite set can have any probability between 0 and 1.

1.3 Consider the set function $\mathbb{P}$ defined for every subset A of $[0, 1]$ by the formula $\mathbb{P}(A) = 0$ if A is a finite set, and $\mathbb{P}(A) = \infty$ if A is an infinite set. Show that $\mathbb{P}$ satisfies the axioms for a probability measure *except* for countable additivity. This shows that the *finite* additivity property does not imply countable additivity.

1.6 Let u be a fixed real number, and define the convex function $f(x) = \mathrm{e}^{ux}$ for all $x \in \mathbb{R}$. Let X be a normal random variable with distribution $\mathrm{N}(\mu, \sigma^2)$. Verify that:

(i) $\mathrm{E}(\mathrm{e}^{uX}) = \exp(u\mu + \frac{1}{2}u^2\sigma^2)$; and that

(ii) Jensen's inequality holds: $\mathrm{E}(f(X)) > f(\mathrm{E}(X))$.

1.11 Consider a standard normal random variable X, which, by the previous exercise has moment-generating function (m.g.f.)

$$\mathrm{E}(\mathrm{e}^{ux}) = \exp u^2/2 \quad \text{for all } u \in \mathbb{R}.$$

The random variable Y was defined to be equal to $X + \theta$, with θ a real constant, and the random variable Z was defined as $Z = \exp(-\theta X - \theta^2/2)$. A new probability measure was defined by

$$\widetilde{\mathbb{P}}(A) = \int_A Z(\omega)\,\mathrm{d}\mathbb{P}(\omega) \quad \text{for all } A \in \mathcal{F}.$$

Prove that Y is standard normal under $\widetilde{\mathbb{P}}$ by showing that the m.g.f. formula

$$\widetilde{\mathrm{E}}(\mathrm{e}^{uY}) = \exp(u^2/2)$$

holds for all $u \in \mathbb{R}$.