Economics 765

May 18, 2024 **R. Davidson**

Assignment 2

You are asked to do exercises 2.3, 2.4, and 2.9 of Volume 2 of Shreve. The essence of these exercises is reproduced below for convenience.

2.3 Let X and Y be independent standard normal random variables. Let θ be a constant, and define the random variables

$$V = X\cos\theta + Y\sin\theta \quad \text{and} \quad W = -X\sin\theta + Y\cos\theta.$$

Show that V and W are independent standard normal variables.

2.4 Let X be a standard normal random variable, and let Z be an independent Rademacher random variable, with

$$P(Z=1) = P(Z=-1) = \tfrac{1}{2}.$$

Let $Y = XZ$. Use moment-generating functions to show that Y is standard normal and that X and Y are not independent, as follows:

(i) Establish the joint m.g.f. of X and Y:

$$\mathrm{E}\mathrm{e}^{uX+vY} = \exp\bigl(\tfrac{1}{2}(u^2+v^2)\bigr)\frac{\mathrm{e}^{uv}+\mathrm{e}^{-uv}}{2}.$$

(ii) Use the formula above to show that $\mathrm{E}\mathrm{e}^{vY} = \mathrm{e}^{v^2/2}$. This is the m.g.f. of a standard normal random variable.

(iii) Use the formula in (i) and Shreve's Theorem 2.2.2(iv) (the fact that independence is equivalent to factorisation of the joint m.g.f) to show that X and Y are not independent.

2.9 Let X be a random variable.

(i) Give an example of a probability space $(\Omega, \mathcal{F}, P)$, a random variable X defined on this probability space, and a function f such that the σ-algebra generated by $f(X)$ is not the trivial σ-algebra $\{\emptyset, \Omega\}$ but is strictly smaller than the σ-algebra generated by X.

(ii) Can the σ-algebra generated by $f(X)$ ever be strictly larger than the σ-algebra generated by X?