Economics 765

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Assignment 3

You are asked to do exercises 3.2, 3.7, and 4.5 of Volume 2 of Shreve. The essence of these exercises is reproduced below for convenience.

3.2 Let $W(t)$, $t \geq 0$, be a Brownian motion, and let $\mathcal{F}(t)$, $t \geq 0$, be a filtration for this Brownian motion. Show that $W^2(t) - t$ is a martingale.

3.7 Theorem 3.6.2 provides the so-called *Laplace transform* of the density of the first passage time for Brownian motion (the moment-generating function). Let W be a Brownian motion. Fix $m > 0$ and $\mu \in \mathbb{R}$. For $0 \leq t < \infty$, define

$$\begin{aligned} X(t) &= \mu t + W(t), \\ \tau_m &= \min\{t \geq 0; X(t) = m\}. \end{aligned}$$

As usual, we set $\tau_m = \infty$ if $X(t)$ never reaches the level m. Let σ be a positive number and set

$$Z(t) = \exp\left\{\sigma X(t) - \left(\sigma\mu + \frac{1}{2}\sigma^2\right)t\right\}.$$

(i) Show that $Z(t)$, $t \geq 0$, is a martingale.

(ii) Use (i) to conclude that

$$\mathrm{E}\left[\exp\left\{\sigma X(t \wedge \tau_m) - \left(\sigma\mu + \frac{1}{2}\sigma^2\right)(t \wedge \tau_m)\right\}\right] = 1, \quad t \geq 0.$$

(iii) Now suppose $\mu \geq 0$. Show that, for $\sigma > 0$,

$$\mathrm{E}\left[\exp\left\{\sigma m - \left(\sigma\mu + \frac{1}{2}\sigma^2\right)\tau_m\right\}\mathrm{I}(\tau_m < \infty)\right] = 1.$$

Use this fact to show that $P\{\tau_m < \infty\} = 1$ and to obtain the Laplace transform

$$\mathrm{E}e^{-\alpha\tau_m} = e^{m\mu - m\sqrt{2\alpha+\mu^2}} \quad \text{for all } \alpha > 0.$$

(iv) Show that, if $\mu > 0$, then $\mathrm{E}\tau_m < \infty$. Obtain a formula for $\mathrm{E}\tau_m$. (Hint: Differentiate the formula in (ii) with respect to α.)

(v) Now suppose $\mu < 0$. Show that, for $\sigma > -2\mu$,

$$\mathrm{E}\left[\exp\left\{\sigma m - \left(\sigma\mu + \frac{1}{2}\sigma^2\right)\tau_m\right\}\mathrm{I}(\tau_m < \infty)\right] = 1.$$

Use this fact to show that $P\{\tau_m < \infty\} = \mathrm{e}^{-2m|\mu|}$ (watch out! there is a misprint here in Shreve, who writes $\mathrm{e}^{-2x|\mu|}$), which is strictly less than 1, and to obtain the Laplace transform

$$\mathrm{E}\mathrm{e}^{-\alpha\tau_m} = \mathrm{e}^{m\mu - m\sqrt{2\alpha+\mu^2}} \quad \text{for all } \alpha > 0.$$

4.5 Let $S(t)$ be a positive stochastic process that satisfies the generalised geometric Brownian motion differential equation

$$\mathrm{d}S(t) = \alpha(t)S(t)\,\mathrm{d}t + \sigma(t)S(t)\,\mathrm{d}W(t),$$

where $\alpha(t)$ and $\sigma(t)$ are processes adapted to the filtration $\mathcal{F}(t)$, $t \geq 0$, associated with the Brownian motion $W(t)$, $t \geq 0$.

(i) Make use of the above differential equation and the Itô-Doeblin formula in order to compute $\mathrm{d}\log S(t)$. Simplify so that you have a formula for $\mathrm{d}\log S(t)$ that does not involve $S(t)$.

(ii) Integrate the formula you obtained in (i), and then exponentiate the answer to obtain the solution

$$S(t) = S(0)\exp\left\{\int_0^t \sigma(s)\,\mathrm{d}W(s) + \int_0^t \left(\alpha(s) - \frac{1}{2}\sigma^2(s)\right)\mathrm{d}s\right\}.$$