Economics 765

June 6, 2024 **R. Davidson**

Assignment 4

You are asked to do exercises 4.19, 5.5, and 5.8 of Volume 2 of Shreve. The essence of these exercises is reproduced below for convenience.

4.19 Let $W(t)$ be a Brownian motion, and define

$$B(t) = \int_0^t \operatorname{sign}(W(s))\,\mathrm{d}W(s),$$

where

$$\operatorname{sign}(x) = \begin{cases} 1 & \text{if } x \geq 0, \\ -1 & \text{if } x < 0. \end{cases}$$

(i) Show that $B(t)$ is a Brownian motion.

(ii) Use Itô's product rule to compute $\mathrm{d}[B(t)W(t)]$. Integrate both sides of the resulting equation and take expectations. Show that $\mathrm{E}[B(t)W(t)] = 0$ (so that $B(t)$ and $W(t)$ are uncorrelated).

(iii) Verify that

$$\mathrm{d}W^2(t) = 2W(t)\,\mathrm{d}W(t) + \mathrm{d}t.$$

(iv) Use Itô's product rule to compute $\mathrm{d}[B(t)W^2(t)]$. Integrate both sides of the resulting equation and take expectations to conclude that

$$\mathrm{E}[B(t)W^2(t)] \neq \mathrm{E}B(t) \cdot \mathrm{E}W^2(t).$$

Explain why this shows that, although they are uncorrelated Gaussian stochastic processes, $B(t)$ and $W(t)$ are not independent.

5.5 You are asked to prove the following result, Corollary 5.3.2 of Shreve.

Let $W(t)$, $0 \leq t \leq T$, be a Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$, and let $\mathcal{F}(t)$, $0 \leq t \leq T$, be the filtration generated by this Brownian motion. Let $\Theta(t)$, $0 \leq t \leq T$, be an adapted process, define

$$Z(t) = \exp\left\{-\int_0^t \Theta(u)\,\mathrm{d}W(u) - \frac{1}{2}\int_0^t \Theta^2(u)\,\mathrm{d}u\right\},$$

$$\widetilde{W}(t) = W(t) + \int_0^t \Theta(u)\,\mathrm{d}u,$$

and assume that $\widetilde{\mathrm{E}} \int_0^T \Theta^2(u) Z^2(u)\,\mathrm{d}u < \infty$. Set $Z = Z(T)$. Then $\mathrm{E}Z = 1$, and under the probability measure $\widetilde{P}$ defined by

$$\widetilde{P}(A) = \int_A Z(\omega)\,\mathrm{d}P(\omega) \quad \text{for all } A \in \mathcal{F},$$

the process $\widetilde{W}(t)$, $0 \le t \le T$, is a Brownian motion.

Now let $\widetilde{M}(t)$, $0 \le t \le T$, be a martingale under $\widetilde{P}$. Then there is an adapted process $\widetilde{\Gamma}(u)$, $0 \le u \le T$, such that

$$\widetilde{M}(t) = \widetilde{M}(0) + \int_0^t \widetilde{\Gamma}(u)\,\mathrm{d}\widetilde{W}(u), \quad 0 \le t \le T. \tag{1}$$

The suggested steps for the proof are as follows.

(i) Compute the differential of $1/Z(t)$.

(ii) Let $\widetilde{M}(t)$, $0 \le t \le T$, be a martingale under $\widetilde{P}$. Show that $M(t) = Z(t)\widetilde{M}(t)$ is a martingale under P.

(iii) According to Shreve's Theorem 5.3.1 (the one-dimensional martingale representation theorem), there is an adapted process $\Gamma(u)$, $0 \le u \le T$, such that

$$M(t) = M(0) + \int_0^t \Gamma(u)\,\mathrm{d}W(u), \quad 0 \le t \le T.$$

Write $\widetilde{M}(t) = M(t)(1/Z(t))$ and take its differential using Itô's product rule.

(iv) Show that the differential of $\widetilde{M}(t)$ is the sum of an adapted process, which we call $\widetilde{\Gamma}(t)$, times $\mathrm{d}\widetilde{W}(t)$, and zero times $\mathrm{d}t$. Integrate to obtain (1).

5.8 (Usual setup and notation.) Assume that there is a unique risk-neutral measure $\widetilde{P}$, and let $\widetilde{W}(t)$, $0 \le t \le T$, be the Brownian motion under $\widetilde{P}$ obtained by using Girsanov's theorem.

Now let $V(T)$ be an almost surely positive $\mathcal{F}(T)$–measurable random variable (under both of the equivalent measures P and $\widetilde{P}$). According to the risk-neutral pricing formula, the price at time t of a security paying $V(T)$ at time T is

$$V(t) = \widetilde{\mathrm{E}}\Big[V(T) \exp - \int_t^T R(u)\,\mathrm{d}u \;\Big|\; \mathcal{F}(t)\Big], \quad 0 \le t \le T.$$

(i) Show that there exists an adapted process $\widetilde{\Gamma}(t)$, $0 \le t \le T$, such that

$$\mathrm{d}V(t) = R(t)V(t)\,\mathrm{d}t + \frac{\widetilde{\Gamma}(t)}{D(t)}\mathrm{d}\widetilde{W}(t), \quad 0 \le t \le T.$$

(ii) Show that, for each $t \in [0, T]$, the price of the derivative security $V(t)$ at time t is almost surely positive.

(iii) Conclude from (i) and (ii) that there exists an adapted process $\sigma(t)$, $0 \leq t \leq T$, such that

$$\mathrm{d}V(t) = R(t)V(t)\,\mathrm{d}t + \sigma(t)V(t)\,\mathrm{d}\widetilde{W}(t), \quad 0 \leq t \leq T.$$

In other words, prior to time T, the price of every asset with almost surely positive price at time T follows a generalised geometric Brownian motion.