June 2024
Final Examination

Models for Financial Economics
Economics 765

Take-home exam due before midnight on Saturday June 22nd 2023.

This exam comprises 5 pages, including the cover page

1. The Black-Scholes-Merton partial differential equation can be written as

$$c_t(t,x) + rxc_x(t,x) + \tfrac{1}{2}\sigma^2 x^2 c_{xx}(t,x) = rc(t,x),$$

where the function $c(t,x)$ is the value at time t of a European option when the price $S(t)$ at t of the underlying asset is x. Subscripts denote partial derivatives. The equation itself applies to any European option, that is, one the payoff of which is a deterministic function of $S(T)$, T being the maturity of the option. Early exercise is not allowed. If the payoff is written as $V(S(T))$, the boundary condition for c is $c(T,x) = V(x)$ for all $x \geq 0$.

Transform the independent variables t and x to two new variables τ and y by the formulas

$$\tau = T - t, \qquad \exp(y) = x\exp\bigl(r - \tfrac{1}{2}\sigma^2\bigr)\tau.$$

Next define a new dependent variable $u(\tau, y) = \mathrm{e}^{r\tau}c(T-t,x)$. Show that u satisfies the partial differential equation

$$u_\tau = \tfrac{1}{2}\sigma^2 u_{yy}.$$

This is a **diffusion equation**. Show that the function

$$u(\tau,y) = \frac{1}{\sigma\sqrt{2\pi\tau}}\exp -\frac{(x-y)^2}{2\sigma^2\tau},$$

x a real number and $\sigma > 0$, satisfies the diffusion equation. Note that this, as a function of y, is just the density function of the normal distribution with expectation x and variance $\sigma^2\tau$. As a function of x, it is the normal density with expectation y and the same variance.

The set of solutions of the diffusion equation constitutes a linear space of functions. Show that the function

$$u(\tau,y) = \frac{1}{\sigma\sqrt{2\pi\tau}}\int_{-\infty}^{\infty} v(x)\exp -\frac{(x-y)^2}{2\sigma^2\tau}\,\mathrm{d}x \qquad (1)$$

is the solution that satisfies the boundary condition $u(0,y) = v(y)$.

Let $V(x) = c(T,x) = (x-K)_+$, the payoff of a European call with strike price K. What does this imply for $v(y) = u(0,y)$? Using your answer to this question, verify that the solution (1) gives the usual Black-Scholes-Merton formula once it is transformed to the original variables.

2. Consider the stopping time τ at which Brownian motion $W(t)$ with $W(0) = 0$ exits the interval $[-a, b]$ of the real line, with $a > 0$ and $b > 0$.

(i) Show that $\tau < \infty$ almost surely.

(ii) What is the probability that it exits the interval at b rather than at $-a$? That is, what is the probability that $W(t)$ hits b before $-a$? (Note that $W(t\wedge\tau)$ is a stopped martingale.)

(iii) Show that $W^2(t\wedge\tau) - t\wedge\tau$ is a martingale.

(iv) Show that $\mathrm{E}(\tau) = ab$.

3. Let P and Q be two equivalent probability measures (they agree as to which sets have measure zero) defined on the measure space $(\Omega, \mathcal{F})$. Let $\mathcal{G}$ be a sub-σ-algebra of $\mathcal{F}$; $\mathcal{G} \subset \mathcal{F}$. We denote the expectations with respect to P and Q as E_P and E_Q respectively. The abstract version of Bayes' Theorem states that, for a random variable X for which $\mathrm{E}_P|X|$ is finite,

$$\mathrm{E}_P(X \,|\, \mathcal{G}) = \frac{\mathrm{E}_Q\Big(\dfrac{\mathrm{d}P}{\mathrm{d}Q} X \;\Big|\; \mathcal{G}\Big)}{\mathrm{E}_Q\Big(\dfrac{\mathrm{d}P}{\mathrm{d}Q} \;\Big|\; \mathcal{G}\Big)}, \tag{2}$$

where as usual $\mathrm{d}P/\mathrm{d}Q$ denotes the Radon-Nikod?m density of the measure P with respect to Q, that is, for all $A \in \mathcal{F}$,

$$P(A) = \mathrm{E}_P(\mathrm{I}_A) = \mathrm{E}_Q\Big(\mathrm{I}_A \frac{\mathrm{d}P}{\mathrm{d}Q}\Big),$$

where I_A is the indicator function for the set A.

The relation (2) can be read as saying that

$$\mathrm{E}_Q\Big(X \,\frac{\mathrm{d}P}{\mathrm{d}Q} \;\Big|\; \mathcal{G}\Big) = \mathrm{E}_P(X \,|\, \mathcal{G})\; \mathrm{E}_Q\Big(\frac{\mathrm{d}P}{\mathrm{d}Q} \;\Big|\; \mathcal{G}\Big).$$

It therefore asserts that the conditional expectation on the left-hand side is given by the right-hand side.

(i) Prove this version of Bayes' Theorem by demonstrating that the right-hand side satisfies the conditions necessary for it to be a version of the conditional expectation on the left-hand side. (The standard machine may be useful here.)

The elementary version of Bayes' Theorem states that, for any $A, B \in \mathcal{F}$,

$$P(A \,|\, B) = \frac{P(A \cap B)}{P(B)}. \tag{3}$$

Let $\sigma(B)$ be the σ-algebra generated by the set B. We have

$$\sigma(B) = \{\emptyset, \Omega, B, B^c\}.$$

We can interpret $P(A \,|\, B)$ as the realisation, for $\omega \in B$, of the conditional expectation $\mathrm{E}_P\big(\mathrm{I}_A \,|\, \sigma(B)\big)$.

(ii) What is the realisation of $E_P\big(\mathrm{I}_A \,|\, \sigma(B)\big)$ for $\omega \in B^c$?

(iii) Show that (3) holds by showing that the random variable

$$Y(\omega) = \begin{cases} P(A \cap B)/P(B) & \text{for } \omega \in B, \\ P(A \cap B^c)/P(B^c) & \text{for } \omega \in B^c. \end{cases}$$

is the expectation of I_A conditional on $\sigma(B)$.

4. This question deals with a variant of the Vasicek interest-rate model.

(i) Find the solution $z(t)$, $t > 0$, of the stochastic differential equation (SDE)

$$\mathrm{d}z(t) = -az(t)\mathrm{d}t + \sigma \mathrm{d}W(t),$$

as a function of the initial condition $z(0)$, where W is a standard Brownian motion, and a and σ are constant positive parameters.

(ii) Find the solution $Y(t)$, $t > 0$, of the SDE

$$\mathrm{d}Y(t) = \big(\theta - aY(t)\big)\mathrm{d}t + \sigma \mathrm{d}W(t)$$

as a function of $Y(0)$, θ being a constant parameter. *Hint:* let $z(t) = Y(t) - \theta/a$.

(iii) Let $x(t) = \exp Y(t)$, $t > 0$. Determine the SDE satisfied by $x(t)$.

(iv) Find the solution of the following interest-rate model

$$\mathrm{d}r(t) = r(t)\big(\eta - a\log r(t)\big)\mathrm{d}t + \sigma r(t)\mathrm{d}W(t),$$

where η is a constant positive parameter, as a function of $r(0)$.

(v) Compute the (unconditional) expectation $\mathrm{E}r(t)$, $t \geq 0$.

(vi) Compute the limit as $t \to \infty$ of $\mathrm{E}r(t)$.

5. The binomial asset pricing model supposes that, in going from period t to period $t+1$, the stock price may move from S_t to S_{t+1} where $S_{t+1} = uS_t$ or $S_{t+1} = dS_t$, with $d < 1+r < u$, where r is the risk-free interest rate. Construct the risk-neutral measure for this model using only the requirement that the discounted stock price must be a martingale under it.

Consider a trinomial asset pricing model, in which, starting from S_t there are three distinct possibilities: S_{t+1} can be uS_t, mS_t, or dS_t, with $d < m < u$ and $d < 1+r < u$.

(i) Does there exist a risk-neutral measure for this model of the stock price, and, if so, is it unique?

(ii) If there is no risk-neutral measure, discover an arbitrage. If there is, and it is not unique, give an example of a derivative security, the random value of which is a deterministic function of the stock price at maturity, that cannot be hedged by a portfolio containing only the stock and a sum of money in the risk-free account with interest rate r. If there is a unique risk-neutral measure, say explicitly how the hedging portfolio is constructed for a European call option.

(iii) In the context of the trinomial model, in which in period $t+1$ there are three possibilities for every one possibility in period t, suppose now that there are two different stocks, each with its own parameters d_i, m_i, u_i, $i = 1, 2$. How are your answers to (i) and (ii) changed in this circumstance?

6. Let $(\Omega, \mathcal{F}, P)$ be a probability space with Ω the unit interval $[0,1]$, $\mathcal{F}$ the Borel σ-algebra $\mathcal{B}$, and P Lebesgue measure. Let $\mathcal{F}_n$ be the σ-algebra generated by the intervals of the form $\big[(j-1)2^{-n}, j2^{-n}\big]$, $j = 1, 2, \ldots, 2^n$, open to the left, closed to the right. Let X be a bounded continuous function on $[0,1]$.

(i) Give the explicit form of the conditional expectation $\mathrm{E}(X \,|\, \mathcal{F}_n)$.

(ii) Show that $\mathcal{F}_n \subset \mathcal{F}_{n+1}$ for all n.

(iii) Show that the sequence of random variables $\{\mathrm{E}(X \,|\, \mathcal{F}_n)\}$, $n = 1, 2, \ldots$, converges for almost all $\omega \in [0,1]$. Characterise the limiting variable. **Hint:** Express ω as the infinite sum

$$\omega = \sum_{j=1}^{\infty} \omega_j 2^{-j}$$

where $\omega_j = 0$ or 1 for all j. Each sequence $\{\omega_j\}$ defines a unique real number ω, and each real $\omega \in [0,1]$ defines a unique sequence, provided that one excludes sequences such that there exists $J > 0$ finite with $\omega_J = 1$ and $\omega_j = 0$ for all $j > J$.

(iv) Show directly that $\mathrm{E}\big(\mathrm{E}(X \,|\, \mathcal{F}_n)\big) = \mathrm{E}(X)$.