Forwards and Futures

A **forward contract** on some asset with time-varying price $S(t)$, $t \in [0,T]$, and strike (delivery) price K is a contract to buy one unit of the asset at time T for price K. The return to the holder of the contract is then $S(T)-K$, which may be positive or negative.

A hedging portfolio for the forward contract contains one unit of the asset and a debt on the money market that, if repaid at time T, is worth exactly K. At time T, this portfolio is worth precisely $S(T)-K$, and so it does correctly hedge the forward contract. The sum borrowed on the money market that leads to a debt of K at time T when the interest rate is r is $\mathrm{e}^{-rT}K$, and so the cost of the hedging portfolio at time 0 is $S(0)-\mathrm{e}^{-rT}K$. To avoid arbitrage, this must also be the cost of the forward contract.

At an intermediate time t, the value of the hedging portfolio is $S(t) - \mathrm{e}^{-r(T-t)}K$. If we want to define a function f of two arguments x and t such that $f\big(S(t),t\big)$ is the value of the hedging portfolio at time t when the asset price is $S(t)$, then it follows at once that $f(x,t) = x - \mathrm{e}^{-r(T-t)}K$.

Suppose that at time 0, an investor wants to find a forward contract that costs nothing now, while obliging him/her to fulfill the terms of the contract by buying a unit of the asset for price K. Such a contract postpones all gains or losses until time T. The appropriate contract must have a **forward price** of K such that the cost $S(0)-\mathrm{e}^{-rT}K=0$, which implies that $K=\mathrm{e}^{rT}S(0)$. At an intermediate time t, the forward price would be recalculated to give a value of $\mathrm{e}^{r(T-t)}S(t)$. We define

$$\mathrm{For}(t,T) = \mathrm{e}^{r(T-t)}S(t). \tag{1}$$

The value of the hedging portfolio at time t is

$$S(t)-\mathrm{e}^{-r(T-t)}K = \mathrm{e}^{-r(T-t)}\big[\mathrm{For}(t,T)-K\big] = \mathrm{e}^{-r(T-t)}\big[\mathrm{For}(t,T)-\mathrm{For}(0,T)\big].$$

If the secondary market for forward contracts is liquid, then this is what the contract with strike K should be worth at time t.

The above assumes a constant interest rate r. For more generality, assume a time-varying discount process $D(t)$. Then, if there exists a risk-neutral measure, the value of the forward contract at time t, $V(t)$, satisfies

$$\begin{aligned} D(t)V(t) &= \widetilde{\mathrm{E}}\big(D(T)(S(T)-K)\mid\mathcal{F}(t)\big) = \widetilde{\mathrm{E}}\big(D(T)S(T)\mid\mathcal{F}(t)\big) - K\widetilde{\mathrm{E}}\big(D(T)\mid\mathcal{F}(t)\big) \\ &= D(t)S(t) - K\widetilde{\mathrm{E}}\big(D(T)\mid\mathcal{F}(t)\big). \end{aligned} \tag{2}$$

This doesn't seem very helpful, since we don't know just what the risk-neutral measure is. However, we get enough information from the bond market in order to evaluate the expression we obtained above.

Consider a zero-coupon bond that will pay one unit of whatever measurement unit we use for the values of assets at time T. Let $B(t,T)$ denote the value of this bond at time t. Then we can evaluate $B(t,T)$ using the risk-neutral measure to get

$$D(t)B(t,T) = \widetilde{\mathrm{E}}\big(D(T).1\mid\mathcal{F}(t)\big) \tag{3}$$

With this, (2) becomes

$$D(t)V(t) = D(t)S(t) - KD(t)B(t,T), \quad \text{or} \quad V(t) = S(t)-KB(t,T).$$

As before, the value of K that sets $V(t)$ equal to zero is the forward price at time t, and so we get

$$\mathrm{For}(t,T) = \frac{S(t)}{B(t,T)}. \tag{4}$$

Check: if $D(t)=\mathrm{e}^{-rt}$, it is deterministic, and (3) gives $B(t,T)=\mathrm{e}^{rt}\mathrm{e}^{-rT}=\mathrm{e}^{-r(T-t)}$, so that $\mathrm{For}(t,T)=\mathrm{e}^{r(T-t)}S(t)$, as before, in (1). Shreve provides another argument, which does not rely on the risk-neutral measure, to show that there is an arbitrage if (4) doesn't hold.

The **futures price** of an asset at time t that matures at time T is defined as

$$\mathrm{Fut}(t,T) = \widetilde{\mathrm{E}}\big(S(T)\mid\mathcal{F}(t)\big).$$

By construction, $\mathrm{Fut}(t,T)$ is a $\widetilde{P}$-martingale (usual argument by iterated conditioning), that satisfies $\mathrm{Fut}(T,T)=S(T)$ and also $\mathrm{Fut}(0,T)=\widetilde{\mathrm{E}}\big(S(T)\big)$. If, but only if, the interest rate is a constant r, the forward and futures price are equal. Indeed,

$$\mathrm{Fut}(t,T) = \mathrm{e}^{rT}\widetilde{\mathrm{E}}\big(\mathrm{e}^{-rT}S(T)\mid\mathcal{F}(t)\big) = \mathrm{e}^{rT}\mathrm{e}^{-rt}S(t) = \mathrm{e}^{r(T-t)}S(t),$$

which, by (1), is $\mathrm{For}(t,T)$ with a constant interest rate. With a time-varying interest rate, there exists the **forward-futures spread**. At time 0, it is $\mathrm{For}(0,T)-\mathrm{Fut}(0,T)$. From (3), we see that $B(0,T)=\widetilde{\mathrm{E}}\big(D(T)\big)$, and so by (4) $\mathrm{For}(0,T)=S(0)/\widetilde{\mathrm{E}}\big(D(T)\big)$, while $\mathrm{Fut}(0,T)=\widetilde{\mathrm{E}}\big(S(T)\big)$. The spread is therefore

$$\begin{aligned} \mathrm{For}(0,T)-\mathrm{Fut}(0,T) &= \frac{S(0)}{\widetilde{\mathrm{E}}(D(T))} - \widetilde{\mathrm{E}}\big(S(T)\big) \\ &= \frac{1}{\widetilde{\mathrm{E}}(D(T))}\big[\widetilde{\mathrm{E}}\big(D(T)S(T)\big) - \widetilde{\mathrm{E}}\big(D(T)\big)\cdot\widetilde{\mathrm{E}}\big(S(T)\big)\big] \\ &= \frac{1}{\widetilde{\mathrm{E}}(D(T))}\widetilde{\mathrm{cov}}\big(S(T),D(T)\big). \end{aligned}$$

A holder of a **futures contract** agrees to receive a cash flow equal to the change in the futures price. If $C(t)$ is the accumulated value of this cash flow at time t, we see that $\mathrm{d}\,C(t) = \mathrm{d}\,\mathrm{Fut}(t,T)$. The cash flow is paid into the

margin account, where it earns or pays interest at the risk-free rate. Let $X(t)$ be the value at time t in the margin account. Then

$$\mathrm{d}X(t) = \mathrm{d}C(t) + R(t)X(t)\mathrm{d}t.$$

Consider the differential of the discounted value. It is

$$\mathrm{d}\big(D(t)X(t)\big) = D(t)\mathrm{d}X(t) + X(t)\mathrm{d}D(t),$$

since $\mathrm{d}D(t)\,\mathrm{d}X(t) = 0$, because $\mathrm{d}D(t) = -R(t)D(t)\,\mathrm{d}t$ has no quadratic variation. Thus

$$\mathrm{d}\big(D(t)X(t)\big) = D(t)\,\mathrm{d}X(t) - R(t)D(t)X(t)\,\mathrm{d}t = D(t)\,\mathrm{d}C(t).$$

On integrating, and noting that $X(0) = 0$ and that $\mathrm{d}C(t) = \mathrm{d}\,\mathrm{Fut}(t,T)$, we see that

$$D(t)X(t) = \int_0^t D(u)\,\mathrm{d}\,\mathrm{Fut}(u,T). \tag{5}$$

Now $\mathrm{Fut}(t,T)$ is a $\widetilde{P}$-martingale, and so also is $D(t)X(t)$, since the integral of an adapted integrand with respect to a martingale is also a martingale (Exercise 4.1 in Shreve). If we think of the margin account as a portfolio, this result is the usual one that the discounted portfolio value is a risk-neutral martingale. This particular result says that, at time $s < t$, the risk-neutral expectation of the change in the value in the margin account in the interval $[s,t]$ is zero. This is why it is costless to abandon the futures contract at any intermediate time, retaining whatever value may be in the margin account.

But if the futures contract is held until maturity, the value in the margin account is determined by setting $t = T$ in (5). We find that

$$D(T)X(T) = \int_0^T D(u)\,\mathrm{d}\,\mathrm{Fut}(u,T).$$

In the case of a constant interest rate, the above expression can be made a little more explicit. Observe that

$$\mathrm{d}\,\mathrm{Fut}(u,T) = \mathrm{d}\,\mathrm{For}(t,T) = \mathrm{e}^{rT}\mathrm{d}(\mathrm{e}^{-rt}S(t))$$

and $D(u) = \mathrm{e}^{-ru}$. Thus

$$D(T)X(T) = \mathrm{e}^{rT}\int_0^T \mathrm{e}^{-ru}\,\mathrm{d}(\mathrm{e}^{-ru}S(u)).$$

We also have

$$D(T)S(T) - S(0) = \int_0^T \mathrm{d}(\mathrm{e}^{-ru}S(u)),$$

and so the discounted value of what the investor effectively has to pay for one unit of the asset at time T is

$$D(T)\big(S(T) - X(T)\big) = S(0) - \int_0^T \big[\mathrm{e}^{r(T-u)} - 1\big]\,\mathrm{d}(\mathrm{e}^{-ru}S(u)).$$